

Systematic effects in the low-energy behaviour of the current SAID solution for the pion-nucleon system

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Abstract

We investigate the description of the pion-nucleon experimental data at low energy (i.e., for pion laboratory kinetic energy $T \leq 100$ MeV) on the basis of the current SAID solution (WI08). We demonstrate that, in a self-consistent analysis scheme, the scale factors of the fits based on the Arndt-Roper formula come out independent of the beam energy and ‘cluster’ around the expectation value of 1. We report systematic effects in regard to the low-energy behaviour of the WI08 solution, indicating that at least one of the assumptions, underlying their analysis framework at all energies (i.e., that the bulk of the data is reliable, that the electromagnetic effects are correctly accounted for, and that isospin invariance is obeyed in the πN system), is not fulfilled.

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1 Introduction

The SAID results for the pion-nucleon (πN) system [1] are widely used as input in numerous works, not only in those studying aspects of the πN interaction. The results represent an optimisation of the description of an extensive database (DB) of measurements, ranging from the πN threshold to (pion laboratory kinetic energy T of) a few GeV. The experimental data is analysed

via dispersion relations. The SAID phase-shift solution is regularly updated, conveniently appearing online, which facilitates the fast dissemination of any new results. Furthermore, new measurements are frequently communicated to the developers of this platform prior to their formal publication, ensuring that the site remain at the state-of-the-art level on a number of hadronic processes. Regarding the πN system, the current SAID solution is dubbed WI08. This paper examines one aspect of this solution, namely its low-energy behaviour. As such, it is expected to be of interest to those who extract important hadronic quantities from the low-energy SAID phase shifts or compare their experimental results to the predictions obtained on the basis of these phase shifts.

Regarding the SAID phase-shift solutions for $T \leq 100$ MeV, our concerns have already been expressed on several occasions. To start with, one has the impression that new πN measurements enter the SAID DB with little regard for their self-consistency and/or compatibility with the measurements which are already present in the DB. Such a strategy would be less problematic, had the SAID group implemented robust statistics in their analysis; however, their results are obtained with a ‘standard’ χ^2 function (i.e., with the Arndt-Roper formula [2]), and are thus expected to be sensitive to the presence of outliers in the DB (in particular, of *one-sided* outliers). Secondly, the SAID results for $T \leq 100$ MeV are literally swamped by the measurements at higher energies. In case that the floating of the data sets is permitted (as the case is when the Arndt-Roper formula is used in the optimisation), it is unavoidable that the low-energy behaviour of the partial-wave amplitudes will be influenced by the measurements acquired at higher energies. Thirdly, the distribution of the normalised residuals, in a self-consistent analysis using a χ^2 minimisation function, must be the normal distribution $N(0, 1)$. Any effects observed in this distribution (e.g., significant offsets, asymmetry, dependence of the normalised residuals on the independent variables in the problem, etc.) are indicative of problems in the input or in its modelling. We are not aware of any paper from the SAID group where these issues are addressed. Regarding the use of the SAID solution for $T \leq 100$ MeV, one final remark is that the measurements from the three πN reactions, i.e., from the two elastic-scattering (ES) reactions ($\pi^\pm p \rightarrow \pi^\pm p$) and from the charge-exchange (CX) reaction ($\pi^- p \rightarrow \pi^0 n$), are forced into an isospin-invariant analysis framework. Evidently, the SAID group choose to disregard the possibility of the violation of the isospin invariance in the hadronic part of the πN interaction, which has been promulgated by some works during the past two decades [3,4,5].

2 Method

2.1 Low-energy parameterisations of the hadronic K -matrix elements

The presumptions in analyses employing the K -matrix parameterisations of this section relate to: a) the number of terms which one retains from the original infinite power series (expansion of the hadronic K -matrix elements in terms of a suitable variable, e.g., of the pion kinetic energy ϵ in the centre-of-mass (CM) system), and b) the forms used in the modelling of the resonant contributions. In both cases, we are confident that our modelling captures the details of the physical system. Such an approach ensures that the detection of any outliers in the input DBs cannot be attributed to the inability of these parametric forms to account for the energy dependence of the hadronic phase shifts; as a result, the detection of outliers in the fits is indicative of experimental discrepancies.

In the analysis of the low-energy πN measurements using the K -matrix parameterisations, we retain terms up to (and including) ϵ^2 . Experience has shown that the coefficients of higher orders in the expansion of the K -matrix elements cannot be determined from the available measurements at low energy.

2.1.1 Fits to our low-energy π^+p database using the K -matrix parameterisations

For π^+p ES, the s -wave phase shift is parameterised as

$$q \cot \delta_{0+}^{3/2} = (a_{0+}^{3/2})^{-1} + b_3 \epsilon + c_3 \epsilon^2, \quad (1)$$

where q denotes (the magnitude of) the CM 3-momentum. The $p_{1/2}$ -wave phase shift is parameterised according to the form

$$\tan \delta_{1-}^{3/2}/q = d_{31} \epsilon + e_{31} \epsilon^2. \quad (2)$$

Since the $p_{3/2}$ wave contains the $\Delta(1232)$ resonance, a singular (at $W = m_\Delta$) term must be added to the background term, leading to the expression

$$\tan \delta_{1+}^{3/2}/q = d_{33} \epsilon + e_{33} \epsilon^2 + \frac{\Gamma_\Delta m_\Delta}{2q_\Delta^3 (p_{0\Delta} + m_p)} \frac{(p_0 + m_p) q^2}{W(m_\Delta - W)}, \quad (3)$$

where Γ_Δ is the $\Delta(1232)$ width, m_Δ is the $\Delta(1232)$ mass, m_p is the proton mass, p_0 is the proton CM energy, and W is the total CM energy. The quantities q_Δ and $p_{0\Delta}$ denote the values of the variables q and p_0 , respectively, at the position of the $\Delta(1232)$ resonance ($W = m_\Delta$). The singular term in Eq. (3)

has been obtained from Ref. [6], see K_{1+} in Eqs. (39) and the corresponding $K_{1+}^{3/2}$ element (after the isospin decomposition of K_{1+} is taken into account), as well as footnote 10 therein.

2.1.2 Fits to our low-energy π^-p elastic-scattering and charge-exchange databases using the K -matrix parameterisations

The isospin $I = 3/2$ amplitudes, obtained in the final fit to the truncated low-energy π^+p DB (i.e., to the DB obtained after the removal of the outliers) using the K -matrix parameterisations of the preceding section, are imported into the analysis of low-energy π^-p ES and CX DBs. In this part, another seven parameters (different for these two DBs) are introduced, to parameterise the $I = 1/2$ amplitudes. The new parametric forms are similar to those given by Eqs. (1-3), with the parameters $a_{0+}^{1/2}$, b_1 , c_1 , d_{13} , e_{13} , d_{11} , and e_{11} . Of course, it is necessary to explicitly include the contribution of the Roper resonance $N(1440)$ in $\delta_{1-}^{1/2}$:

$$\tan \delta_{1-}^{1/2}/q = d_{11}\epsilon + e_{11}\epsilon^2 + \frac{\Gamma_R M_R(p_{0R} + m_p)}{2q_R^3(M_R + m_p)^2} \frac{(W + m_p)^2 q^2}{W(M_R - W)(p_0 + m_p)} \quad , \quad (4)$$

where (as we are dealing with energies below the pion-production threshold) Γ_R is the partial width of the Roper resonance to πN decay modes and m_R is its mass. The quantities q_R and p_{0R} denote the q and p_0 values at the resonance position ($W = M_R$). The singular term in Eq. (4) has been obtained from Ref. [6], see Section 3.5.1 therein, in particular, Eq. (54) for K_{1-} .

2.2 Minimisation function

In our recent partial-wave analyses (PWAs) of the πN data, we make use of the minimisation function given by the Arndt-Roper formula [2], i.e., of the minimisation function which the SAID group also use in their analyses. The contribution of the j -th data set to the overall χ^2 is of the form:

$$\chi_j^2 = \sum_{i=1}^{N_j} \left(\frac{z_j y_{ij}^{\text{th}} - y_{ij}^{\text{exp}}}{\delta y_{ij}^{\text{exp}}} \right)^2 + \left(\frac{z_j - 1}{\delta z_j} \right)^2 \quad , \quad (5)$$

where y_{ij}^{exp} denotes the i -th data point of the j -th data set, y_{ij}^{th} the corresponding fitted (‘theoretical’) value, $\delta y_{ij}^{\text{exp}}$ the statistical uncertainty of the y_{ij}^{exp} data point, z_j a scale factor applied to the entire data set, δz_j the normalisation uncertainty (reported by the experimental group or assigned by us), and N_j the number of the data points in the data set after the removal of any outliers. The fitted values y_{ij}^{th} are obtained by means of the parameterised forms of the s - and p -wave amplitudes detailed in Sections 2.1.1 and 2.1.2. The values of

the scale factor z_j are determined (separately for each data set) in such a way as to minimise χ_j^2 . For each data set, a unique solution for z_j is obtained via the relation:

$$z_j = \frac{\sum_{i=1}^{N_j} y_{ij}^{\text{th}} y_{ij}^{\text{exp}} / (\delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}} , \quad (6)$$

which leads to

$$(\chi_j^2)_{\min} = \sum_{i=1}^{N_j} \frac{(y_{ij}^{\text{th}} - y_{ij}^{\text{exp}})^2}{(\delta y_{ij}^{\text{exp}})^2} - \frac{\left(\sum_{i=1}^{N_j} y_{ij}^{\text{th}} (y_{ij}^{\text{th}} - y_{ij}^{\text{exp}}) / (\delta y_{ij}^{\text{exp}})^2 \right)^2}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}} . \quad (7)$$

The overall $\chi^2 = \sum_{j=1}^N (\chi_j^2)_{\min}$ (where N denotes the number of the accepted data sets in the fit) is a function of the parameters entering the modelling of the s - and p -wave amplitudes. These parameters are varied until χ^2 attains its minimal value χ_{\min}^2 .

The part of $(\chi_j^2)_{\min}$ which represents the pure random fluctuation in the measurements of the j -th data set (i.e., the ‘unexplained variation’ in standard regression terminology) may be obtained from Eq. (7) in the limit $\delta z_j \rightarrow \infty$, which is equivalent to removing the term $(\delta z_j)^{-2}$ from the denominator of the second term on the right-hand side (rhs) of the expression; we denote this value by $(\chi_j^2)_{st}$. The variation which is contained in $(\chi_j^2)_{\min}$ in excess of $(\chi_j^2)_{st}$ is associated with the contribution from the floating (rescaling) of the data set. The expression for $(\chi_j^2)_{sc} \equiv (\chi_j^2)_{\min} - (\chi_j^2)_{st}$ is

$$(\chi_j^2)_{sc} = \frac{(\delta z_j)^{-2} \left(\sum_{i=1}^{N_j} y_{ij}^{\text{th}} (y_{ij}^{\text{th}} - y_{ij}^{\text{exp}}) / (\delta y_{ij}^{\text{exp}})^2 \right)^2}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 \left(\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2} \right)} . \quad (8)$$

The scale factors which minimise only the first term on the rhs of Eq. (5) are obtained from Eq. (6) in the limit $\delta z_j \rightarrow \infty$:

$$\hat{z}_j = \frac{\sum_{i=1}^{N_j} y_{ij}^{\text{th}} y_{ij}^{\text{exp}} / (\delta y_{ij}^{\text{exp}})^2}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2} . \quad (9)$$

The scale factors \hat{z}_j represent the optimal floating of the fitted values y_{ij}^{th} around the experimental results y_{ij}^{exp} , with no regard to the normalisation uncertainty.

The analysis of this work may be performed using either of the two scale factors, z_j or \hat{z}_j . We make use of the scale factors z_j , as only these quantities are contained in the output of the SAID Analysis Program.

For the purpose of the optimisation, we employ the standard MINUIT package [7] of the CERN library (FORTRAN version). Each optimisation has been achieved on the basis of the (robust) SIMPLEX-MINIMIZE-MIGRAD-MINOS sequence. All fits of this work terminated successfully.

3 Results from our partial-wave analysis

The list of the experiments, which are included in our three πN DBs, is available from Refs. [5,8]; the same notation will be used here to identify the individual data sets. Our DBs comprise differential cross sections (DCSs), analysing powers (APs), partial-total cross sections (PTCSs), and total (as well as total-nuclear) cross sections (TCSs) for $T \leq 100$ MeV. The largest difference between the SAID DB for $T \leq 100$ MeV and ours pertains to the ES DCSs of the CHAOS Collaboration [9,10]. Regarding these measurements, our opinion is known [11,12] and there is no reason to repeat it here.

Modifications in the analysis software and DB structure in the recent years enable us now to also include in the optimisation the APs of Ref. [13], comprising a total of 28 data points. There had been two technical reasons preventing the direct use of these measurements in our PWAs before 2013: a) each of the three data sets, to which the measurements of Ref. [13] must be assigned, involves more than one beam energies and b) the last of these data sets contains measurements of both ES reactions.

Regarding the proton electromagnetic (EM) form factors, recent developments suggest the replacement of the forms we had been using before 2013. The parameterisation of the Dirac $F_1^p(t)$ and Pauli $F_2^p(t)$ form factors of the proton with (traditional) dipole forms has been found to provide a poor description of the ‘world’ electron-proton (ep) unpolarised and polarised data [14]. Although the sensitivity of our results to the details of the parameterisation of these quantities is low (due to the smallness of the Q^2 transfer for $T \leq 100$ MeV), adopted now is an improved parameterisation. In Ref. [15], the authors made use of the so-called Padé parameterisation [16] for the Sachs EM form factors $G_E^p(t)$ and $G_M^p(t)$ (in Ref. [15], the superscript p is omitted), and obtained the optimal values of the relevant parameters from a fit to ep measurements; we now use the results of their Table II. The pion form factor $F^\pi(t)$ is usually parameterised via a monopole form, e.g., see Ref. [17]. Although results of the same quality are obtained in the low-energy region with either a monopole or a dipole form, the monopole parameterisation is now adopted.

The values of the relevant physical constants (see Table 1) have been fixed from the most recent compilation of the Particle-Data Group (PDG) [18]. Finally, an improved approach for determining the (small) d and f waves has been implemented; to suppress artefacts which are due to the truncation of small values, simple polynomials are now fitted to the d - and f -wave phase shifts of the SAID analysis [1], which (as it incorporates dispersion-relation constraints) is expected to determine reliably these phase shifts in the region $T > 100$ MeV.

One statistical test for each data set is performed, the one involving its contribution $(\chi_j^2)_{\min}$ to the overall χ_{\min}^2 (see Section 2.2). The p-value, estimated from $(\chi_j^2)_{\min}$ and N_j , is compared to the user-defined confidence level p_{\min} for the acceptance of the null hypothesis (no statistically significant effects); in case that the extracted p-value is below p_{\min} , the degree of freedom (DOF) with the largest contribution to $(\chi_j^2)_{\min}$ is eliminated in the subsequent fit. As in our recent PWAs, we adopt the p_{\min} value which is associated with a 2.5σ effect in the normal distribution. This value is approximately equal to $1.24 \cdot 10^{-2}$, i.e., slightly larger than $1.00 \cdot 10^{-2}$, a threshold which most statisticians recommend as the outset of statistical significance.

When identifying the outliers in our past PWAs, the maximal number of excluded data points for each data set had been fixed (somewhat arbitrarily) at 2; data sets with more excluded DOFs were removed from the analysis. To do justice to data sets containing a large number of data points, this restriction was recently revised¹. Assuming pure statistical fluctuation, the probability that the j -th data set contains at most 2 outliers decreases with increasing N_j values. Evidently, the maximal number of outliers must be determined separately for each data set, on the basis of the number of measurements which the data set initially contains.

Let us assume that the *a-priori* probability for a data point to be an outlier is equal to p . The probability that the j -th data set (with N_j initial data points) contains exactly k outliers is then given by the expression

$$P_k = \binom{N_j}{k} p^k (1-p)^{N_j-k} . \quad (10)$$

Consequently, the probability that the j -th data set contains up to $N_j^{\text{out}} \leq N_j$ outliers is given by the sum

$$P(N_j^{\text{out}}) = \sum_{k=0}^{N_j^{\text{out}}} P_k . \quad (11)$$

After p_{\min} is set, the maximal number of outliers permitted for the particular experiment (at that p_{\min} level) may be identified as the maximal N_j^{out} value for which the cumulative probability, obtained with Eq. (11), does not exceed $1 - p_{\min}$. It thus follows that, if $p = 0.06$, a data set with 20 data points will be allowed to contain 3 outliers, whereas one with only 10 data points a maximum of 2 outliers for the p_{\min} value adopted herein. To obtain an estimate of p , the maximal number of outliers N_j^{out} was set equal to N_j (which is equivalent to relaxing the condition for removing entire data sets) and the fits to our low-energy π^+p DB (which is known to contain the largest proportion of outliers)

¹ This modification does not affect any of the results we have obtained thus far for $p_{\min} \approx 1.24 \cdot 10^{-2}$.

were carried out iteratively, excluding at each iteration step the data point with the largest contribution to the overall χ^2_{\min} . The procedure was repeated until no data point could be identified as an outlier. An estimate of the probability p is obtained as the fraction of the number of DOFs (NDF) of the initial π^+p DB which had to be eliminated. To avoid the exclusion of low- N_j data sets just because of the removal of 1 DOF, N_j^{out} was finally redefined as $\max\{N_j^{\text{out}}, 2\}$.

Data sets which do not give acceptable p -values (i.e., exceeding p_{\min}) after the elimination of the appropriate number of data points (the absolute normalisation is also considered to be one of the acquired measurements), as explained above, were removed from the DB. Only one point was removed at each step. The optimisation was repeated, until no data point could be identified as an outlier. It must be emphasised that, in our approach, the identification of the outliers in one reaction (e.g., in the π^-p ES DB) is based on comparisons involving only the measurements in that particular reaction.

3.1 Fits to our low-energy π^+p database

Our initial low-energy π^+p DB contains 57 data sets consisting of 389 data points. As in all our past PWAs, we found that the data sets of BRACK90 at 66.80 MeV and JORAM95 at 32.70 MeV (with 11 and 7 data points, respectively) had to be removed from the DB. Subsequently, three data sets had to be freely floated, namely two of the BRACK86 data sets (at 66.80 and 86.80 MeV), as well as the BRACK90 data set at 30.00 MeV. Three additional single data points had to be removed. After the elimination of 24 DOFs of the initial DB, we obtained a truncated low-energy π^+p DB comprising 55 data sets and 365 DOFs. The accepted data sets are detailed in Table 2.

The χ^2_{\min} value, corresponding to the fit to the initial DB, was 706.7; for the truncated DB, $\chi^2_{\min} \approx 459.5$. Therefore, the removal of 24 DOFs from the initial DB leads to the decrease of the χ^2 by 247.2 units, corresponding to more than 10 units per removed entry on average. At the same time, the p -value of the fit increased by over 17 orders of magnitude.

The scale factors z_j , corresponding to the DCSs in our low-energy π^+p DB ², are shown in Fig. 1. The weighted linear fit to the data shown (the weight of each entry is equal to $(\delta z_j)^{-2}$) yields an intercept of 1.023 ± 0.033 and a slope of $(-0.38 \pm 0.40) \cdot 10^{-3} \text{ MeV}^{-1}$; both fitted values are compatible with the expectation values (of 1 and 0, respectively) for a successful optimisation.

² Being ratios of cross sections, the APs are not suitable for the demonstration of the systematic effects which we have set about investigating in this study. The PTCSs and the TCSs (all one- or two-point data sets) could have been used in the comparison, yet the SAID group do not include these measurements in their DB.

3.2 *Fits to our low-energy π^-p elastic-scattering database*

Our initial low-energy π^-p ES DB contains 37 data sets consisting of 339 data points. The BRACK90 data set at 66.80 MeV (with 5 data points in total) needed to be removed, one data set (the WIEDNER89 data set at 54.30 MeV) had to be freely floated, and two additional single data points had to be removed. After the elimination of 8 DOFs of the initial DB, we obtained a truncated low-energy π^-p ES DB comprising 36 data sets and 331 DOFs. The accepted data sets are detailed in Table 3.

The χ^2_{\min} value, corresponding to the fit to the initial DB, was 524.8; for the truncated DB, $\chi^2_{\min} \approx 371.0$. Therefore, the removal of only 8 DOFs from the initial DB leads to a decrease of the χ^2 by 153.8 units, i.e., just short of 20 units per removed entry on average. At the same time, the p-value of the fit increased by over 8 orders of magnitude.

The scale factors z_j , corresponding to the DCSs in our low-energy π^-p ES DB, are shown in Fig. 2. The weighted linear fit to the data shown yields an intercept of 1.007 ± 0.019 and a slope of $(-0.08 \pm 0.24) \cdot 10^{-3} \text{ MeV}^{-1}$; both fitted values are compatible with the expectation for a successful optimisation.

3.3 *Fits to our low-energy π^-p charge-exchange database*

Our initial low-energy π^-p CX DB contains 54 data sets consisting of 333 data points. Only 5 DOFs needed to be removed: the FITZGERALD86 data sets at 32.48, 36.11, 40.26, and 47.93 MeV needed to be freely floated, and the BREITSCHOPF06 TCS at 75.10 MeV had to be removed. After the elimination of these 5 DOFs of the initial DB, we obtained a truncated low-energy π^-p CX DB comprising 53 data sets and 328 DOFs. The accepted data sets are detailed in Table 4.

The χ^2_{\min} value, corresponding to the fit to the initial DB, was 401.6; for the truncated DB, $\chi^2_{\min} \approx 313.7$. Therefore, the removal of only 5 DOFs from the initial DB leads to a decrease of the χ^2 by 87.9 units, i.e., to 17.6 units per removed entry on average.

The scale factors z_j , corresponding to the DCSs in our low-energy π^-p CX DB, are shown in Fig. 3. The weighted linear fit to the data shown yields an intercept of 1.002 ± 0.015 and a slope of $(0.05 \pm 0.27) \cdot 10^{-3} \text{ MeV}^{-1}$; again, both fitted values are compatible with the expectation for a successful optimisation.

4 Results from the WI08 solution

The SAID π^+p DB for $T \leq 100$ MeV contains 721 data points, 679 of which relate to DCSs, the remaining 42 to APs; PTCSs and TCSs are not included. The DB contains the entirety of the DENZ05 data, as well as the BERTIN76 measurements (save for the 67.40 MeV data set). The BERTIN76, the AULD79, and the FRANK83 data sets are floated ³. The overall χ^2_{\min} , corresponding to the SAID π^+p DB for $T \leq 100$ MeV, is equal to 1595.2, yielding a p-value of a few 10^{-68} . Evidently, the description of the SAID π^+p DB for $T \leq 100$ MeV with the WI08 phase-shift solution is poor.

The SAID π^-p ES DB for $T \leq 100$ MeV contains 634 data points, 546 of which relate to DCSs, the remaining 88 to APs. Similar to us (but for different reasons ⁴), they have not included the corresponding PTCSs and TCSs in their DB. Their DB contains the entirety of the DENZ05 data. The four FRANK83 data sets are floated. The overall χ^2_{\min} , corresponding to the SAID π^-p ES DB for $T \leq 100$ MeV, is equal to 1102.1, yielding a p-value of about 10^{-27} . Therefore, the description of the SAID π^-p ES DB for $T \leq 100$ MeV with the WI08 solution is almost as poor as that of their π^+p DB.

The SAID π^-p CX DB for $T \leq 100$ MeV contains 353 data points, 343 of which relate to DCSs, the remaining 10 to APs; the corresponding TCSs are not included therein. The three DUCLOS73 measurements have been deleted from their DB. The overall χ^2_{\min} , corresponding to the SAID π^-p CX DB for $T \leq 100$ MeV, is equal to 401.3, yielding a p-value of about $3.9 \cdot 10^{-2}$. The description of the SAID π^-p CX DB for $T \leq 100$ MeV with the WI08 solution is acceptable.

There are three reasons why the description of *their* low-energy ES DBs with *their* WI08 solution is poor.

- The SAID πN DB is very extensive. The low-energy behaviour of the partial-wave amplitudes is largely determined from measurements acquired at higher energies. As the total amount of the measurements, contained in

³ The definition of floating for the SAID group is not identical to ours. For us, free floating involves the limit $\delta z_j \rightarrow \infty$, whereas they simply set the corresponding normalisation uncertainty δz_j of the data set to the value of 1. This value significantly reduces, but does not eliminate, the contribution of the floating contribution $(\chi^2_j)_{sc}$ of Eq. (8) to $(\chi^2_j)_{\min}$ of Eq. (7).

⁴ Given that the nine available π^-p PTCSs and TCSs contain a component from CX scattering, these measurements have never been included in our π^-p ES DB. Their involvement in any part of the analysis (see beginning of Section 5) would perplex the discussion on the violation of the isospin invariance in the hadronic part of the πN interaction.

their DB, exceeds 50 000 data points, their 1 708 low-energy measurements are literally swamped in the analysis.

- The SAID group include in their DBs all the available measurements (including a few data sets which have not appeared in formal publications); outliers are seldom excluded.
- The three πN reactions are subject to an analysis which assumes the fulfillment of the isospin invariance in the πN system at all energies.

We will now report the results of a very simple analysis of the scale factors z_j for $T \leq 100$ MeV, as these quantities come out of their fits to the data. These values (two decimal digits are available online) were obtained from the SAID web page on May 6, 2016; we also use their T_j and δz_j values. Scatter plots of the scale factors z_j versus the corresponding beam energy T_j for the data sets of the SAID DB are shown, *separately* for the three πN reactions, in Figs. 4-6. The fitted values of the parameters of the weighted linear fits (as well as their uncertainties, corrected with the Birge factor $\sqrt{\chi^2/\text{NDF}}$), are shown in Table 5. Visual inspection of these figures and of Table 5 leads to two conclusions:

- In all three πN reactions, the departure of the scale factors z_j from the expectation for a successful optimisation is noticeable. The energy dependence of the scale factors z_j is more pronounced in the case of the $\pi^+ p$ reaction.
- The departure from the expectations appears to occur slightly below $T = 100$ MeV; decreasing effects with increasing beam energy are observed.

If the scale factors from the three πN reactions are analysed in a joint scheme, then the systematic effects, observed in the individual reactions, disappear. This becomes evident after comparing the last row of Table 5 with the previous three, as well as after comparing Fig. 7 with Figs. 4-6.

5 Similarities and differences between the two approaches

In our method of extracting information from the πN measurements for $T \leq 100$ MeV, three steps are followed. At the first step, we employ simple parameterisations of the s - and p -wave K -matrix elements, retaining in the expressions orders up to ϵ^2 . We make use of these general parameterisations in order to reliably identify and remove any outliers present in the initial DBs, thus obtaining self-consistent input for the subsequent phases of the analysis. At the second step of our approach, the ETH model of the πN interaction (a complete description of this model and details on its development may be found in Ref. [6]) is fitted to the truncated DBs of the two ES reactions, thus leading to the determination of the values of the model parameters (coupling constants and vertex factors), which account optimally for the measurements of these two reactions. At a third step, we investigate the violation of the

isospin invariance in the πN system, by comparing the model predictions for the CX reaction (obtained from the fitted values, as well as the correlation matrix of the fit to the two ES DBs) to the measurements of the CX reaction. We reported in the past that these predictions significantly *underestimate* the CX DCSs in most of the low-energy region. Assuming the correctness of the bulk of the available measurements and the smallness of any missing pieces in the EM corrections, this mismatch between predictions and data strongly indicates the violation of the isospin invariance in the πN system at low energy (for a more detailed discussion, see Section 7 of Ref. [5]). Following a slightly different methodology (and fewer CX measurements), Gibbs, Ai, and Kaufmann came to the same conclusion in the mid 1990s [3].

Our PWAs are restricted to $T \leq 100$ MeV, because the ETH model is expected to work better in the low-energy region and because our EM corrections [19,20] have been established only below 100 MeV. Regarding the former remark, the introduction of strong-interaction form factors in the Feynman graphs of the ETH model is unnecessary below 100 MeV and the contributions from graphs involving distant baryonic states - i.e., the higher baryon resonances with masses above 2 GeV - to the s - and p -wave amplitudes of the model are negligible.

A dispersion-relation analysis framework, such as the one developed by the SAID group, assumes the fulfillment of the isospin invariance in the πN system and analyses the entire DB available, extending to T values of several GeV, so that the dispersion integrals be evaluated reliably. Such an approach cannot be performed in a restricted energy region, without bringing in external influences. Such a framework cannot be used in order to conduct exclusive analyses of the data, be they restricted in energy or involving only one reaction. This is one important difference between our approach and theirs. The second difference relates to the procedure: we choose to analyse the two ES processes (this analysis fixes the $I = 3/2$ and $I = 1/2$ amplitudes) and subsequently investigate whether isospin invariance is fulfilled, on the basis of comparisons with the measurements of the third reaction (triangle identity). On the contrary, the SAID group assume the fulfillment of the isospin invariance in the πN system at all energies, and analyse the three πN reactions in a joint optimisation scheme.

So far, the emphasis has been placed on the differences between the two approaches. However, there are also similarities. For instance, both approaches rely on the correctness of the input data. To an extent, we take steps towards investigating the self-consistency of the input DB, whereas they avoid excluding measurements. Only considering the number of the input data points, the sensitivity of our analysis to outliers is expected to be more pronounced than it is in their case: 500 discrepant DCSs in a DB of 1 000 data points wreck havoc; the same amount of discrepant measurements in a DB of over 50 000

data points is surely less of a problem. As a result, we cannot but thoroughly examine every new data set prior to including it in the DB; in this respect, they can afford to be more generous. Finally, both approaches rely on the correctness/completeness of the method incorporating the EM effects (distortion corrections to the hadronic phase shifts and to the partial-wave amplitudes) in the analysis.

To summarise, the SAID group assume that isospin invariance is fulfilled at all energies, whereas we test whether it is at low energy. They cannot easily restrict their analysis to specific energy regions or reactions, whereas we may perform a variety of analyses in relation to the input DBs: we may perform joint analyses (e.g., by using as input the π^+p DB along with either of the π^-p DBs or by submitting the measurements of all three πN reactions to a joint optimisation scheme) or we may analyse the three πN reactions independently of one another (using our K -matrix parameterisations). However, our analyses can only involve the low-energy region. Both approaches rely on the correctness of the bulk of the measurements (ours is significantly more sensitive to the presence of outliers in the DB) and both approaches rely on the correctness of the EM corrections, applied to the hadronic phase shifts and to the partial-wave amplitudes on the way to fitting the observables.

6 Discussion and conclusions

A self-consistent, χ^2 -based optimisation scheme satisfies the requirement that the distribution of the normalised residuals of the fit be normal, i.e., that they follow the $N(0, 1)$ distribution. Furthermore, these residuals must not exhibit significant dependences on the independent variables in the problem, e.g., on the beam energy in this work. In a self-consistent optimisation scheme, where the input measurements are reliable and their modelling adequate, all fluctuations present in the residuals are random. We demonstrated that the results of the weighted linear fits to the scale factors z_j , obtained from the data when using our K -matrix parameterisations, come out as expected: independent of the beam energy and ‘clustering’ around the expectation value of 1 (see Figs. 1-3).

The SAID analysis rests upon the fulfillment of three conditions *at all energies*:

- (a) that the bulk of the experimental data is reliable;
- (b) that the EM effects are correctly accounted for;
- (c) and that the isospin invariance is fulfilled in the πN system.

When the Arndt-Roper formula is used in the optimisation, one expects that the data sets which must be scaled ‘upwards’ should be balanced (on average)

by those which must be scaled ‘downwards’. Additionally, the energy dependence of the scale factors must not be significant. If these prerequisites are not fulfilled, the modelling of the input measurements cannot be considered as successful. The scale factors, relating to the description of the SAID low-energy πN DBs with the WI08 solution, are shown in Fig. 7. At first glance, the plot leaves a satisfactory impression, in particular after considering the extent of the SAID πN DB, as well as the practice of the SAID group to avoid excluding the outliers. Consequently, the optimisation scheme leading to the WI08 solution yields normalised residuals for the scale factors ⁵ which are reasonably well centred on 0 (we disregard small effects). Given that the reduced χ^2 , corresponding to the entirety of the fitted data at low energy in the case of the WI08 solution, is about 1.81, the joint analysis of the data inevitably yields normalised residuals with a distribution broader than the normal. However, this broadening originates in the choice of the SAID group to seldom exclude outliers.

Let us finally express our criticism on the low-energy behaviour of the WI08 solution. The SAID input DB comprises three distinct parts, identified as the sets of measurements of the three πN reactions. Had their fit been unbiased, the general behaviour of the fitted z_j values in terms of their dependence on T_j , as obtained from Fig. 7, would also have been observed in any arbitrary subset of their DB, consistent with the basic principles of the Sampling Theory (adequate population, representative sampling). The scale factors, relating to the description of the SAID low-energy πN DBs with the WI08 solution, are shown (separately for the three reactions) in Figs. 4-6. Had the three aforementioned conditions (a)-(c) been fulfilled in the SAID analysis at all energies, the scale factors of Figs. 4-6 would have come out independent of the beam energy and would have been centred on 1, as the case was for the joint analysis of Fig. 7. However, the bulk of the data for $T \leq 100$ MeV (represented by the shaded bands in Figs. 4-6) appears to be either underestimated by the WI08 solution (i.e., in case of the CX reaction) or overestimated by it (i.e., in case of the two ES reactions, the effects for the π^+p reaction being more pronounced). One notices that the mismatches decrease with increasing beam energy, converging to $z = 1$ in the vicinity of $T = 100$ MeV. Such a behaviour is consistent with the conclusions of Refs. [3,4,5] for an energy-dependent isospin-breaking effect.

From Figs. 4-6 and from Table 5, one may conclude that the WI08 solution does not describe sufficiently well the bulk of the low-energy measurements in any of the three πN reactions. Evidently, the WI08 solution at low energy represents a fictitious, average πN process, one which does not adequately capture the dynamics of the three physical πN reactions. Regarding the analysis of the scale factors obtained with the WI08 solution, we stress (once again)

⁵ These residuals are equal to $(z_j - 1)/\delta z_j$, see Eq. (5).

that their z_j values, as they appear in the SAID web page [1], have only been fitted to by us with straight lines. Of course, no input from our approach is used in this part of the analysis.

The conclusion appears to be inevitable. Even in a framework of an analysis assuming the isospin invariance (as the case is for the WI08 solution), the isospin-breaking effects, albeit somewhat hidden, manifest themselves as systematic trends in the output of the optimisation and may be uncovered easily if the three πN reactions are analysed *separately*. This conclusion corroborates our insistence during the past two decades on investigating the behaviour of the normalised residuals in our PWAs of the low-energy measurements.

One might argue that the effects, contained in Figs. 4-6 and in Table 5, are not ‘large’. However, it must be borne in mind that the isospin-breaking effects are not expected to be large. Gibbs, Ai, and Kaufmann [3] have shown that such effects mainly affect the s -wave part of the amplitude (10% effects are seen in their Fig. 1), and (to a lesser extent) the no-spin-flip p -wave part (smaller effects are seen in their Figs. 2). We have shown that the largest departure from the isospin invariance occurs in the phase shifts $\delta_{0+}^{1/2}$ and $\delta_{1+}^{3/2}$, see Figs. 2 and 3 of Ref. [5]. Both analyses appear to agree on the expected magnitude of such effects: roughly speaking, effects between 5 and 10% (in the amplitude) are expected at low energy. The systematic effects, observed in Figs. 4-6, have about the right size, as well as an energy dependence which is consistent with the findings of Refs. [3,4,5].

The effects, shown in Figs. 4-6 and in Table 5, are systematic and are indicative of the non-fulfillment of at least one of the three conditions (a)-(c) listed in the beginning of this section. One of the safest conclusions, drawn from Figs. 1-3, is that our approach yields results which closely represent and reproduce the bulk of the measurements for $T \leq 100$ MeV; the low-energy behaviour of the WI08 solution does not appear to properly account for the bulk of the low-energy data as successfully. If future experimental activity in the πN sector reveals that, on average, the meson-factory, low-energy experiments had been affected by severe energy-dependent systematic effects, then a solution based on a dispersion-relation analysis framework (like WI08) may be more reliable than ours. We will not speculate on how such effects could affect the two ES processes in one way (i.e., resulting in a systematic underestimation of the relevant DCSs) and the CX reaction in another (i.e., resulting in a systematic overestimation of the relevant DCSs). If, on the other hand, the persistent discrepancies, observed in the πN system at low energy, are to be blamed elsewhere (i.e., on a departure from a theoretical constraint), then our choice to test the isospin invariance, rather than assume it, will be proven justified.

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Table 1

The current values of the physical constants, used in this analysis. These values have been taken from the most recent compilation of the Particle-Data Group [18]. Regarding the Roper resonance $N(1440)$, the partial width Γ_R of Section 2.1.2 is the product of the total width Γ_T and the corresponding branching ratio η for the πN decay modes of the resonance.

Physical quantity (unit)	Value
Charged-pion mass (MeV)	139.57018
Proton mass m_p (MeV)	938.272046
$\Delta(1232)$ mass m_Δ (MeV)	1232
$\Delta(1232)$ decay width Γ_Δ (MeV)	117
$N(1440)$ M_R (MeV)	1440
$N(1440)$ Γ_T (MeV)	300
$N(1440)$ η	0.650

Table 2

The data sets comprising the truncated π^+p database, the pion laboratory kinetic energy T_j of the data set (in MeV), the number of degrees of freedom N_j after the removal of the outliers, the scale factor z_j which minimises χ_j^2 of Eq. (5), the normalisation uncertainty δz_j (reported by the experimental group or, if not reported, assigned by us, e.g., as the case is for the AULD79 data set), the value of $(\chi_j^2)_{\min}$, and the p-value of the fit. In the case of free floating, z_j is equal to \hat{z}_j of Eq. (9). The numbers in this table correspond to the final fit to the data using the K -matrix parameterisations of Section 2.1.1.

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Differential cross sections							
AULD79	47.90	11	1.0290	0.1097	16.4530	0.1251	
RITCHIE83	65.00	8	1.0469	0.0240	18.2215	0.0196	
RITCHIE83	72.50	10	1.0062	0.0200	4.7931	0.9046	
RITCHIE83	80.00	10	1.0289	0.0140	18.9749	0.0406	
RITCHIE83	95.00	10	1.0308	0.0150	12.1585	0.2746	
FRANK83	29.40	28	1.0423	0.0370	19.3186	0.8880	
FRANK83	49.50	28	1.0581	0.2030	34.4110	0.1877	
FRANK83	69.60	27	0.9304	0.0950	23.2937	0.6691	
FRANK83	89.60	27	0.8603	0.0470	29.1001	0.3561	
BRACK86	66.80	4	0.8941	0.0120	2.4056	0.6616	freely floated
BRACK86	86.80	8	0.9368	0.0140	16.3766	0.0373	freely floated
BRACK86	91.70	5	0.9726	0.0120	12.5697	0.0278	
BRACK86	97.90	5	0.9709	0.0150	7.6323	0.1777	
BRACK88	66.80	6	0.9488	0.0210	10.6388	0.1002	
BRACK88	66.80	6	0.9574	0.0210	9.3760	0.1535	
WIEDNER89	54.30	19	0.9894	0.0304	14.8601	0.7314	
BRACK90	30.00	5	1.1805	0.0360	8.4943	0.1310	freely floated
BRACK90	45.00	8	1.0258	0.0220	8.4449	0.3913	
BRACK95	87.10	8	0.9717	0.0220	13.8636	0.0854	
BRACK95	98.10	8	0.9796	0.0200	15.0499	0.0582	
JORAM95	45.10	9	0.9712	0.0330	17.8659	0.0368	124.42° removed
JORAM95	68.60	9	1.0516	0.0440	9.1437	0.4241	
JORAM95	32.20	20	1.0224	0.0340	31.2306	0.0522	
JORAM95	44.60	18	0.9621	0.0340	27.6309	0.0679	30.74°, 35.40° removed

Table 2 continued

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Analysing powers							
SEVIOR89	98.00	6	1.0128	0.0740	5.5312	0.4777	
WIESER96	68.34	3	0.9100	0.0500	4.5009	0.2122	
WIESER96	68.34	4	0.9316	0.0500	4.7278	0.3164	
MEIER04	57.20-87.20	12	0.9824	0.0350	13.9222	0.3057	
MEIER04	45.20, 51.20	6	0.9709	0.0350	7.4638	0.2801	
MEIER04	57.30-87.20	7	1.0038	0.0350	11.0607	0.1360	
Partial-total cross sections							
KRISS97	39.80	1	1.0144	0.0300	2.5847	0.1079	
KRISS97	40.50	1	1.0023	0.0300	0.2646	0.6070	
KRISS97	44.70	1	1.0042	0.0300	0.1939	0.6597	
KRISS97	45.30	1	1.0053	0.0300	0.2402	0.6241	
KRISS97	51.10	1	1.0265	0.0300	4.1742	0.0410	
KRISS97	51.70	1	1.0046	0.0300	0.1507	0.6979	
KRISS97	54.80	1	1.0111	0.0300	0.3650	0.5457	
KRISS97	59.30	1	1.0294	0.0300	1.7134	0.1905	
KRISS97	66.30	2	1.0523	0.0300	4.4518	0.1080	
KRISS97	66.80	2	1.0077	0.0300	0.6370	0.7272	
KRISS97	80.00	1	1.0135	0.0300	0.3320	0.5645	
KRISS97	89.30	1	1.0074	0.0300	0.2509	0.6164	
KRISS97	99.20	1	1.0541	0.0300	3.9747	0.0462	
FRIEDMAN99	45.00	1	1.0467	0.0600	2.7169	0.0993	
FRIEDMAN99	52.10	1	1.0217	0.0600	0.3985	0.5279	
FRIEDMAN99	63.10	1	1.0401	0.0600	0.5992	0.4389	
FRIEDMAN99	67.45	2	1.0546	0.0600	1.3394	0.5119	
FRIEDMAN99	71.50	2	1.0508	0.0600	0.8682	0.6479	
FRIEDMAN99	92.50	2	1.0411	0.0600	0.5389	0.7638	

Table 2 continued

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Total-nuclear cross sections							
CARTER71	71.60	1	1.0940	0.0600	2.7880	0.0950	
CARTER71	97.40	1	1.0475	0.0600	0.6326	0.4264	
PEDRONI78	72.50	1	1.0126	0.0600	0.1451	0.7033	
PEDRONI78	84.80	1	1.0302	0.0600	0.3085	0.5786	
PEDRONI78	95.10	1	1.0215	0.0600	0.1766	0.6743	
PEDRONI78	96.90	1	1.0154	0.0600	0.1132	0.7365	

Table 3

The equivalent of Table 2 for the fits to our π^-p elastic-scattering database. The numbers of this table correspond to the final fit to the data using the K -matrix parameterisations of Section 2.1.2.

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Differential cross sections							
FRANK83	29.40	28	0.9841	0.0350	30.9946	0.3173	15.55° removed, freely floated
FRANK83	49.50	28	1.0985	0.0780	29.4176	0.3916	
FRANK83	69.60	27	1.0921	0.2530	24.7552	0.5882	
FRANK83	89.60	27	0.9457	0.1390	24.8611	0.5822	
BRACK86	66.80	5	0.9974	0.0130	13.9614	0.0159	
BRACK86	86.80	5	1.0031	0.0120	1.3828	0.9262	
BRACK86	91.70	5	0.9962	0.0120	3.0724	0.6888	
BRACK86	97.90	5	0.9998	0.0120	5.9607	0.3101	
WIEDNER89	54.30	18	1.1567	0.0304	23.5424	0.1706	
BRACK90	30.00	5	1.0210	0.0200	5.0513	0.4096	
BRACK90	45.00	9	1.0512	0.0220	11.6945	0.2311	36.70° removed
BRACK95	87.50	6	0.9812	0.0220	10.6655	0.0993	
BRACK95	98.10	7	1.0072	0.0210	8.4645	0.2934	
JORAM95	32.70	4	0.9941	0.0330	3.9286	0.4158	
JORAM95	32.70	2	0.9533	0.0330	5.6499	0.0593	
JORAM95	45.10	4	0.9540	0.0330	12.5594	0.0136	
JORAM95	45.10	3	0.9450	0.0330	9.5406	0.0229	
JORAM95	68.60	7	1.0829	0.0440	14.3284	0.0456	
JORAM95	68.60	3	1.0322	0.0440	2.3004	0.5124	
JORAM95	32.20	20	1.0615	0.0340	21.3464	0.3770	
JORAM95	44.60	20	0.9434	0.0340	30.3198	0.0648	
JANOUSCH97	43.60	1	1.0395	0.1500	0.1482	0.7002	
JANOUSCH97	50.30	1	1.0320	0.1500	0.1189	0.7302	
JANOUSCH97	57.30	1	1.0829	0.1500	4.7450	0.0294	
JANOUSCH97	64.50	1	1.0212	0.1500	0.0298	0.8629	
JANOUSCH97	72.00	1	1.2988	0.1500	4.6476	0.0311	

Table 3 continued

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Analysing powers							
ALDER83	98.00	6	1.0103	0.0400	5.3078	0.5050	
SEVIOR89	98.00	5	0.9871	0.0740	1.7012	0.8887	
HOFMAN98	86.80	11	1.0019	0.0300	5.9941	0.8738	
PATTERSON02	57.20	10	0.9468	0.0370	10.5993	0.3896	
PATTERSON02	66.90	9	0.9988	0.0370	4.6096	0.8669	
PATTERSON02	66.90	10	0.9522	0.0370	16.4217	0.0882	
PATTERSON02	87.20	11	0.9835	0.0370	8.1202	0.7025	
PATTERSON02	87.20	11	0.9944	0.0370	4.9386	0.9341	
PATTERSON02	98.00	12	0.9950	0.0370	6.5960	0.8831	
MEIER04	67.30, 87.20	3	0.9929	0.0350	3.2264	0.3580	

Table 4

The equivalent of Table 2 for the fits to our π^-p charge-exchange database. The numbers of this table correspond to the final fit to the data using the K -matrix parameterisations of Section 2.1.2.

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Differential cross sections							
DUCLOS73	22.60	1	0.9422	0.0800	1.2111	0.2711	
DUCLOS73	32.90	1	0.9715	0.0800	0.2741	0.6006	
DUCLOS73	42.60	1	0.9100	0.0800	2.3763	0.1232	
FITZGERALD86	32.48	2	1.5008	0.0780	2.3360	0.3110	freely floated
FITZGERALD86	36.11	2	1.7161	0.0780	1.2144	0.5449	freely floated
FITZGERALD86	40.26	2	1.8332	0.0780	6.5440	0.0379	freely floated
FITZGERALD86	47.93	2	1.4509	0.0780	1.6502	0.4382	freely floated
FITZGERALD86	51.78	3	1.1216	0.0780	7.2398	0.0646	
FITZGERALD86	55.58	3	1.0933	0.0780	2.5533	0.4657	
FITZGERALD86	63.21	3	1.0548	0.0780	1.3372	0.7203	
FRLEZ98	27.50	6	1.0904	0.0870	10.4326	0.1076	
ISENHOWER99	10.60	4	1.0196	0.0600	2.1628	0.7058	
ISENHOWER99	10.60	5	1.0054	0.0400	1.4569	0.9180	
ISENHOWER99	10.60	6	1.0176	0.0400	8.0670	0.2332	
ISENHOWER99	20.60	5	0.9799	0.0400	1.5691	0.9050	
ISENHOWER99	20.60	6	1.0120	0.0400	8.1842	0.2249	
ISENHOWER99	39.40	4	1.0707	0.0600	7.3129	0.1202	
ISENHOWER99	39.40	5	1.0593	0.0400	8.4695	0.1322	
ISENHOWER99	39.40	5	0.9521	0.0400	5.1290	0.4003	
SADLER04	63.86	20	0.9540	0.0650	16.2394	0.7017	
SADLER04	83.49	20	0.9877	0.0520	11.7013	0.9260	
SADLER04	94.57	20	1.0303	0.0450	7.2943	0.9956	
JIA08	34.37	4	0.8433	0.1000	4.9400	0.2935	
JIA08	39.95	4	0.8676	0.1000	3.1953	0.5257	
JIA08	43.39	4	0.8780	0.1000	2.5106	0.6427	
JIA08	46.99	4	0.9799	0.1000	5.1602	0.2713	

Table 4 continued

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Differential cross sections							
JIA08	54.19	4	0.9057	0.1000	2.1191	0.7139	
JIA08	59.68	4	0.9318	0.1000	3.0789	0.5447	
MEKTEROVIC09	33.89	20	1.0239	0.0340	17.0108	0.6523	
MEKTEROVIC09	39.38	20	1.0145	0.0260	14.7679	0.7895	
MEKTEROVIC09	44.49	20	1.0100	0.0270	33.1582	0.0324	
MEKTEROVIC09	51.16	20	1.0355	0.0290	15.0640	0.7727	
MEKTEROVIC09	57.41	20	1.0390	0.0290	19.4998	0.4896	
MEKTEROVIC09	66.79	20	1.0227	0.0300	19.5380	0.4871	
MEKTEROVIC09	86.62	20	1.0016	0.0290	31.0086	0.0551	
Legendre expansion of the differential cross section							
SALOMON84	27.40	3	0.9722	0.0310	2.7669	0.4290	
SALOMON84	39.30	3	0.9939	0.0310	1.0926	0.7789	
BAGHERI88	45.60	3	1.0055	0.0310	0.1372	0.9870	
BAGHERI88	62.20	3	0.9582	0.0310	3.6125	0.3065	
BAGHERI88	76.40	3	0.9725	0.0310	3.3190	0.3450	
BAGHERI88	91.70	3	1.0151	0.0310	2.8312	0.4184	
Measurement of the width of pionic hydrogen							
SCHROEDER01	0.00	1	0.9746	0.0225	2.5283	0.1118	
Analysing powers							
STASKO93	100.00	4	0.9944	0.0440	1.4508	0.8353	
GAULARD99	98.10	6	1.0234	0.0450	1.0835	0.9823	

Table 4 continued

Data set	T_j	N_j	z_j	δz_j	$(\chi_j^2)_{\min}$	p-value	Comments
Total cross sections							
BUGG71	90.90	1	1.0226	0.0600	0.1478	0.7007	
BREITSCHOPF06	38.90	1	0.9960	0.0300	0.1641	0.6854	
BREITSCHOPF06	43.00	1	1.0011	0.0300	0.0259	0.8722	
BREITSCHOPF06	47.10	1	0.9980	0.0300	0.0578	0.8100	
BREITSCHOPF06	55.60	1	0.9951	0.0300	0.2131	0.6444	
BREITSCHOPF06	64.30	1	0.9724	0.0300	3.8191	0.0507	
BREITSCHOPF06	65.90	1	0.9777	0.0300	2.3820	0.1227	
BREITSCHOPF06	76.10	1	0.9812	0.0300	1.6410	0.2002	
BREITSCHOPF06	96.50	1	0.9819	0.0300	0.5924	0.4415	

Table 5

The fitted values of the parameters of the weighted linear fit to the data shown in Figs. 4-7, as well as their uncertainties, corrected with the Birge factor $\sqrt{\chi^2/\text{NDF}}$, taking account of the goodness of each fit.

Reaction	Intercept	Slope (10^{-3} MeV^{-1})
π^+p	0.936 ± 0.028	0.70 ± 0.34
π^-p elastic-scattering	0.972 ± 0.021	0.16 ± 0.27
π^-p charge-exchange	1.041 ± 0.019	-0.43 ± 0.34
All three πN reactions	0.985 ± 0.014	0.10 ± 0.18

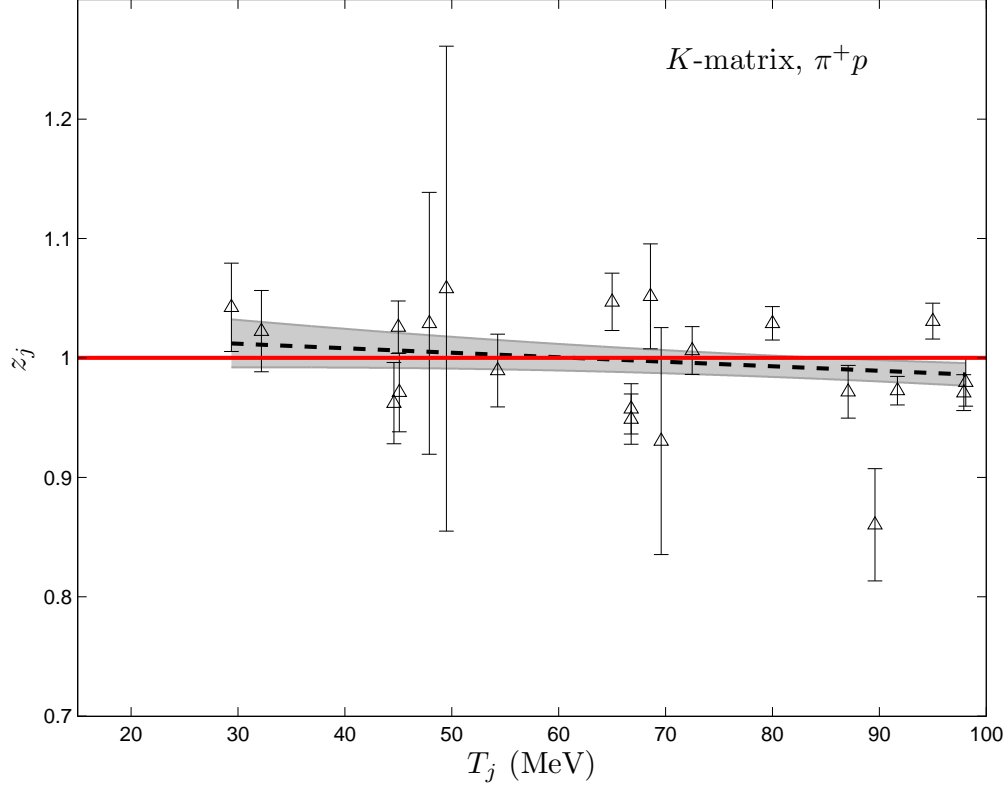


Fig. 1. Plot of the scale factors z_j which minimise χ_j^2 of Eq. (5) for the fits to our low-energy π^+p database using the K -matrix parameterisations of Section 2.1.1; T_j denotes the pion laboratory kinetic energy of the j -th experiment. The data correspond to DCSs only; the data sets which had to be freely floated are not shown. The dashed straight line represents the optimal, weighted linear fit to the data shown and the shaded band 1σ uncertainties around the fitted values. The red line is the optimal, unbiased outcome of the optimisation.

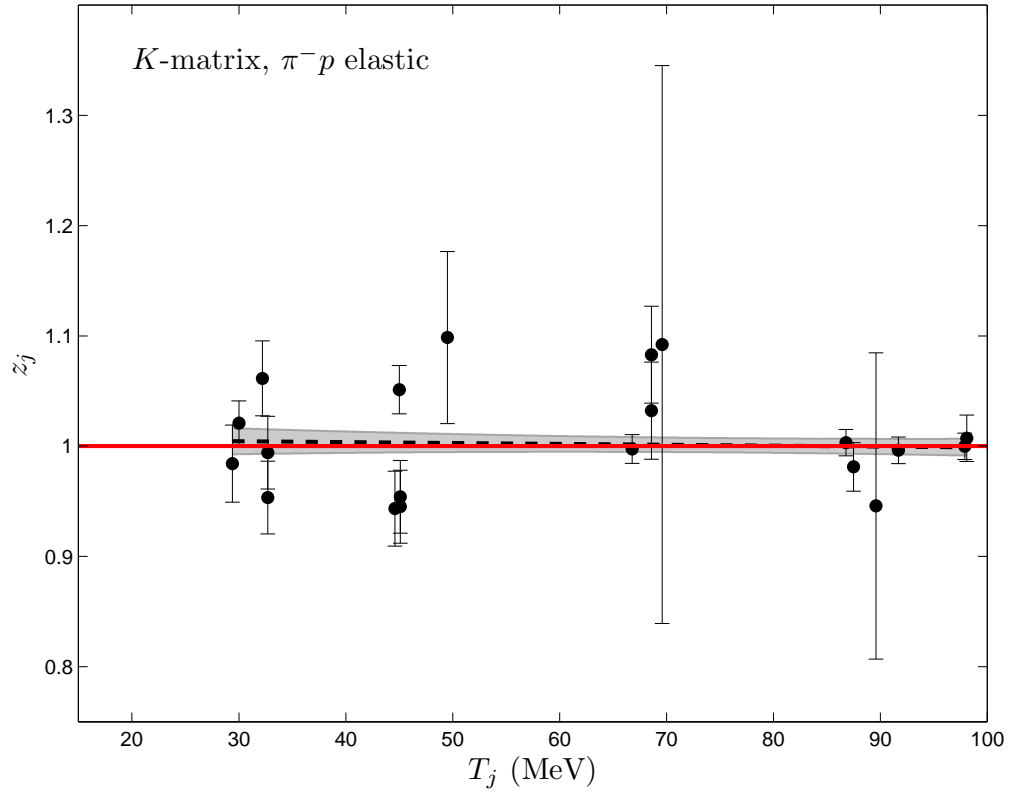


Fig. 2. The equivalent of Fig. 1 for the fits to our π^-p elastic-scattering database.

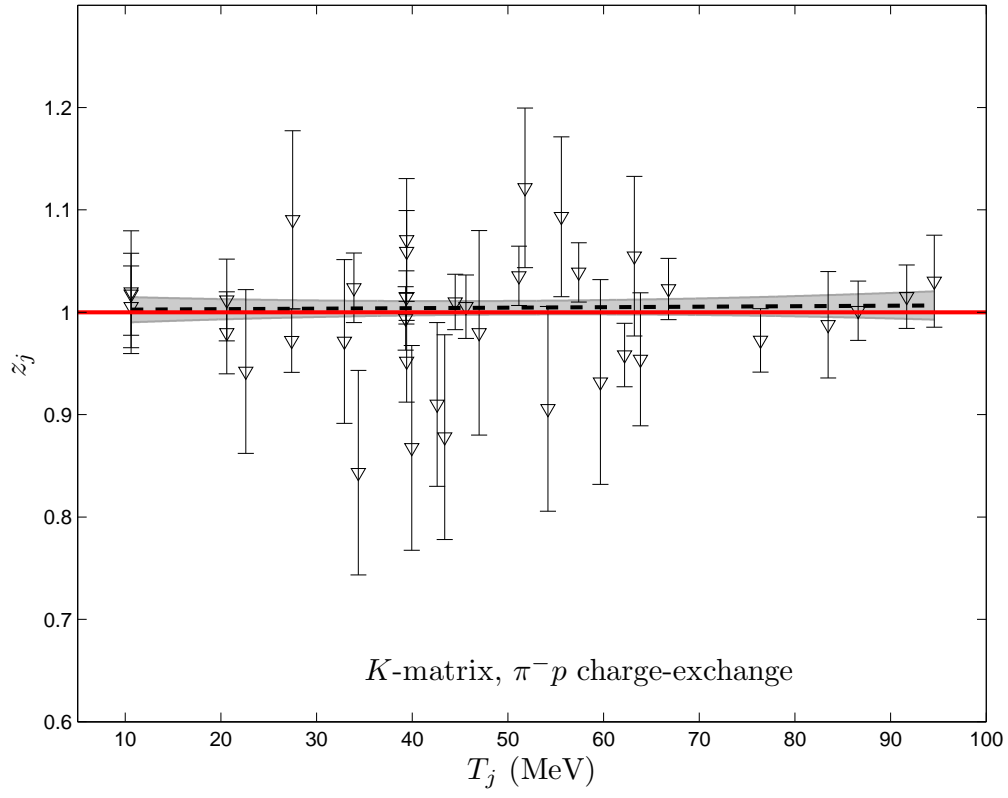


Fig. 3. The equivalent of Fig. 1 for the fits to our $\pi^- p$ charge-exchange database.

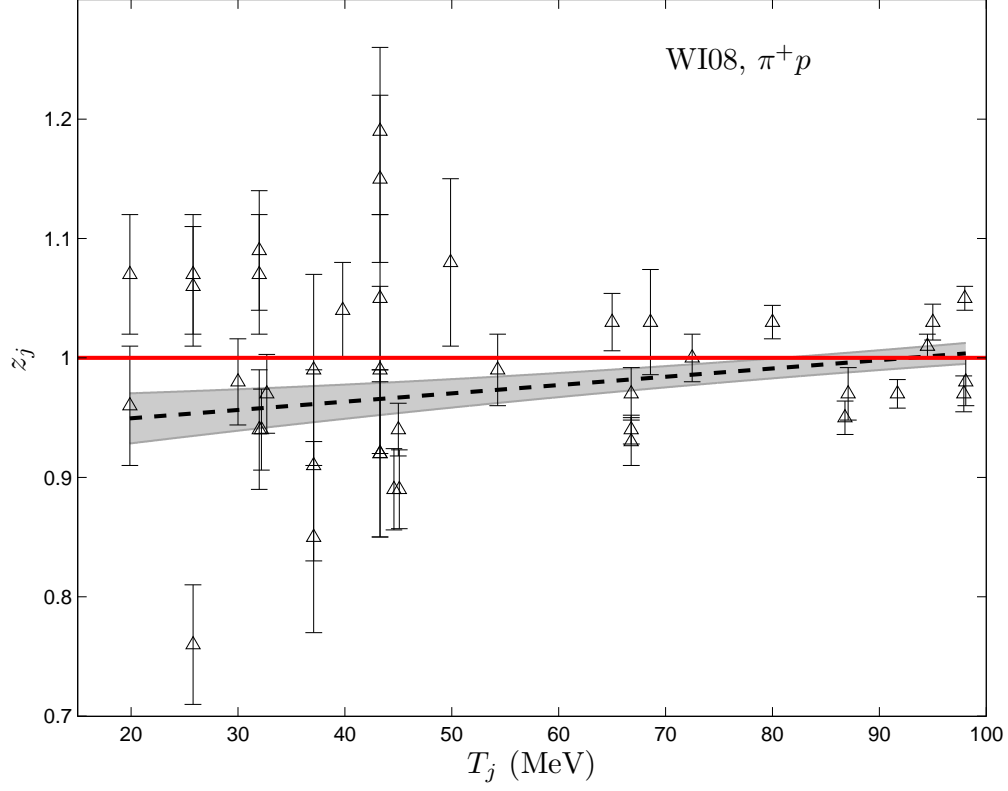


Fig. 4. Plot of the scale factors z_j which minimise χ_j^2 of Eq. (5) for the fits to the SAID low-energy π^+p database (yielding the WI08 solution); T_j denotes the pion laboratory kinetic energy of the j -th experiment. The data correspond to DCSs only; the data sets which were floated are not shown. The dashed straight line represents the optimal, weighted linear fit to the data shown (see Table 5) and the shaded band 1σ uncertainties around the fitted values. The red line is the optimal, unbiased outcome of the optimisation.

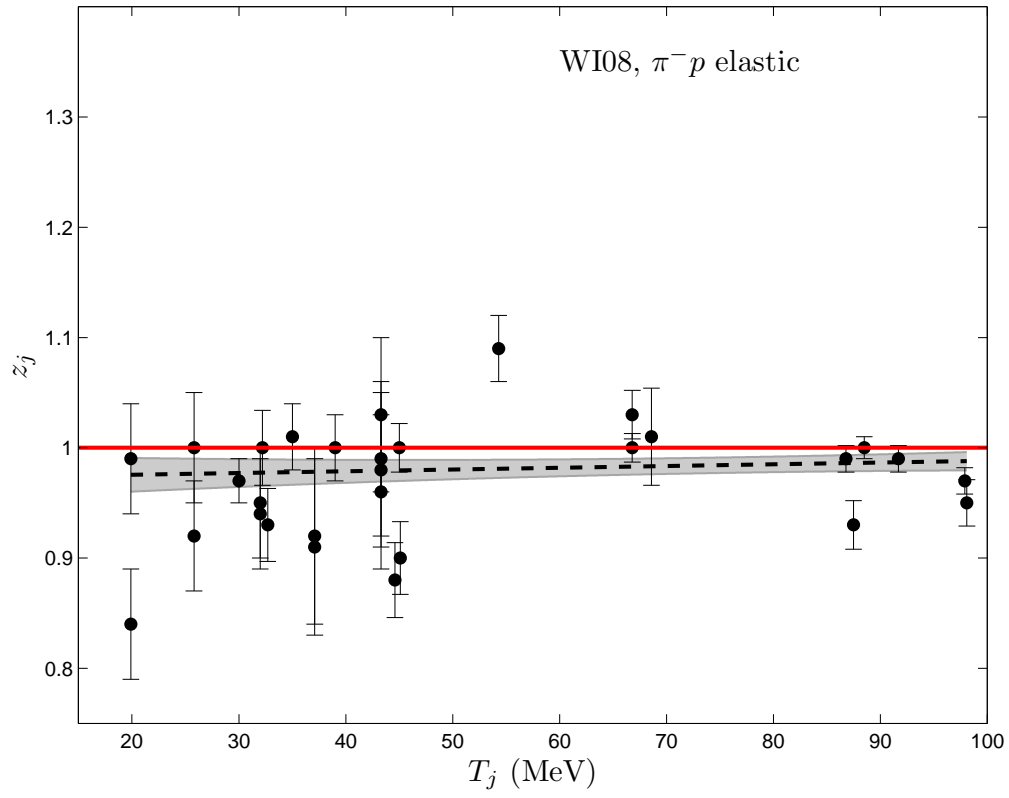


Fig. 5. The equivalent of Fig. 4 for the SAID low-energy π^-p elastic-scattering database.

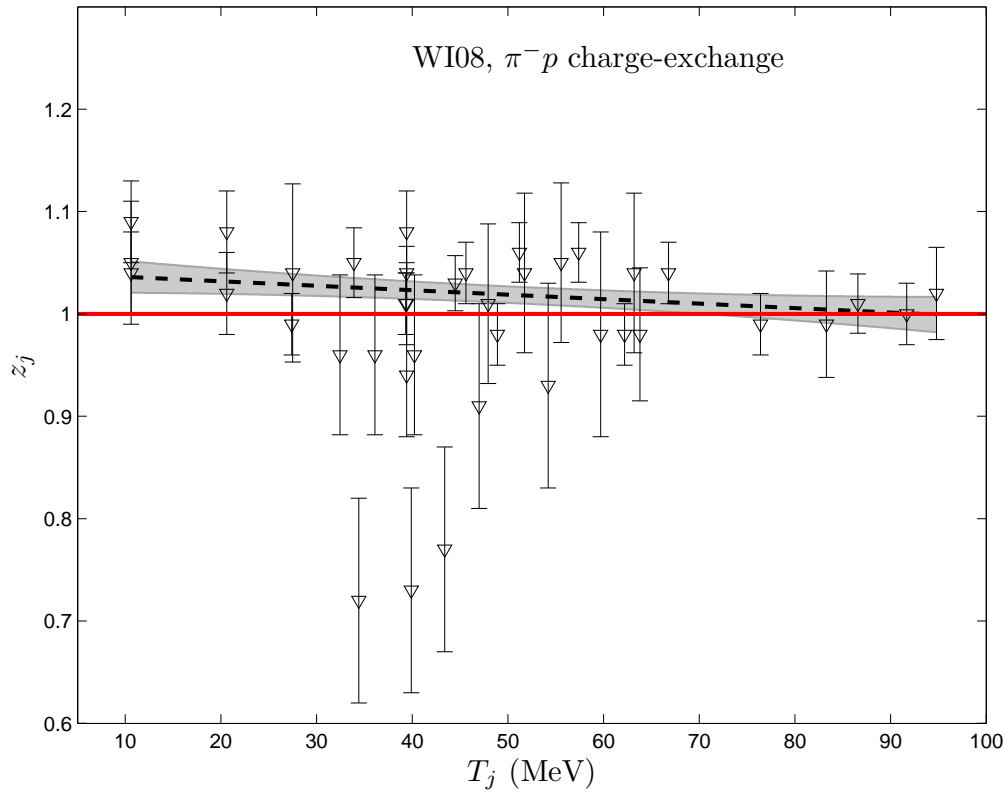


Fig. 6. The equivalent of Fig. 4 for the SAID low-energy $\pi^- p$ charge-exchange database.

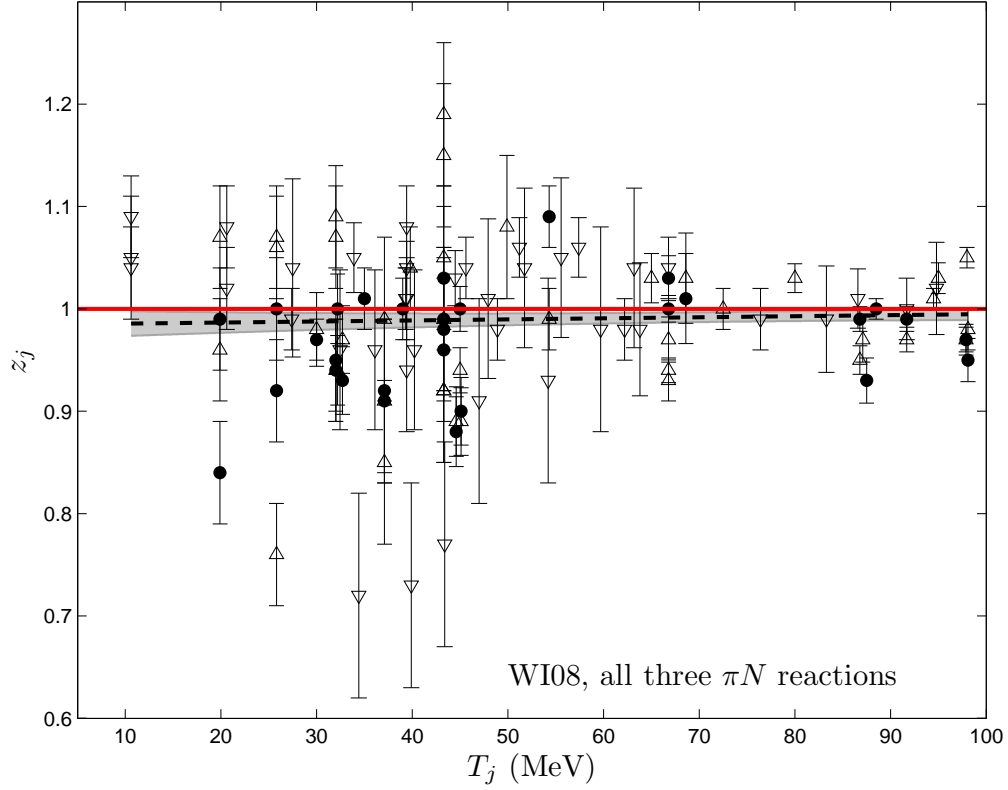


Fig. 7. Plot of the scale factors z_j which minimise χ_j^2 of Eq. (5) for the fits to the entire SAID low-energy πN database (yielding the WI08 solution); T_j denotes the pion laboratory kinetic energy of the j -th experiment. The data correspond to DCSs only (upward triangles: π^+p , dots: π^-p elastic, downward triangles: π^-p charge-exchange); the data sets which were floated are not shown. The dashed straight line represents the optimal, weighted linear fit to the data shown (see Table 5) and the shaded band 1σ uncertainties around the fitted values. The red line is the optimal, unbiased outcome of the optimisation.